Uniform growth of groups acting on Cartan-Hadamard spaces.

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1 Introduction

In this paper we investigate the growth of finitely generated groups. Given a group Γ generated by a finite set S, the word length $l_S(\gamma)$ of an element $\gamma \in \Gamma$ is the smallest integer m such that there exist elements $\sigma_1, \ldots, \sigma_m$ in $S \cup S^{-1}$ with $\gamma = \sigma_1 \ldots \sigma_m$. The entropy of Γ with respect to the generating set S is defined by

$$\mathrm{Ent}_S(\Gamma) = \lim_{m\to\infty} \frac{1}{m}(\log |\{\gamma \in \Gamma \,/\, l_S(\gamma) \leq m\}|)\,. \tag{1}$$

If $\mathrm{Ent}_S(\Gamma) > 0$ for some generating set S, it is true for all (finite) generating set and the group is said to have exponential growth. We now define the entropy of Γ

$$\mathrm{Ent}\,\Gamma = \inf_S \{\mathrm{Ent}_S(\Gamma) \,/\, S \,\text{finite generating set of}\, \Gamma\}\,. \tag{2}$$

We say that Γ has uniform exponential growth if $\mathrm{Ent}\,\Gamma > 0$. In [11], remarque 5.12, M. Gromov raised the question whether exponential growth always implies uniform exponential growth. The answer is negative, indeed, in [14] J.S. Wilson gave examples of finitely generated groups of exponential growth and non uniform exponential growth. Nevertheless, exponential growth implies uniform exponential growth for hyperbolic groups [12], geometrically finite groups of isometries of Hadamard manifolds with pinched negative curvature [1], solvable groups [13] and linear groups [10], [4], [3]. For further references see the exposition paper [7].

We suppose that (X, g) is a n-dimensional Cartan Hadamard manifold of pinched sectional curvature $-a^2 \leq K \leq -1$. Our main result is the

Theorem 1.1 *There exists a positive constant $C(n, a)$ such that for any finitely generated discrete group Γ of isometries of (X, g), then either Γ is virtually nilpotent or $\mathrm{Ent}(\Gamma) \geq C(n, a)$.*

Remark. The difficulty is here to show that one can choose the constant $C(n,a)$ not depending on the group Γ. In the linear setting, E. Breuillard obtained the same kind of uniformity proving the existence of a positive constant $C(n)$ such that for any finitely generated group Γ of $GL(n,K)$, K any field, then either Γ is virtually solvable or $\mathrm{Ent}(\Gamma) \geq C(n)$.

The classical technique is to prove that "not too far" from any finite generating system one can exhibit a free group (in two generators). In this paper we do prove this in one of the cases under consideration, using the famous ping-pong lemma, however in the second case we use a different approach using natural Lipschitz maps from the Cayley graph into X. This is the new idea which is described in the following.

In a private communication M. Kapovich mentioned to us a different proof in the case when Γ acts without any elliptic element. One important issue in our proof is that we do not have this restriction, elliptic elements are permitted.

In the forthcoming paper [2] we shall use this result to prove a Margulis lemma without curvature; indeed, we shall replace the curvature assumptions by hypothesis on the growth of the fundamental group.

2 Preliminaries

Let (X,g) be a n-dimensional Cartan-Hadamard manifold with sectional curvature $-a^2 \leq K_g \leq -1$. Let us recall a few well-known facts about isometries. If γ is an isometry of (X,g), the displacement of γ is defined by $l(\gamma) = \inf_{x\in X} \rho(x,\gamma x)$, where ρ is the distance associated to the metric g on X. We then have (see [9] p. 31):

1. The isometry γ is called hyperbolic (or axial) if $l(\gamma) > 0$, in which case there exists a geodesic a_γ, called the axis of γ, such that, for any $x \in a_\gamma$, $\rho(x,\gamma x) = l(\gamma)$.

2. The isometry is γ is called parabolic if $l(\gamma) = 0$ and $l(\gamma)$ is not achieved on X, in which case there exists a unique point θ on the geometric boundary ∂X of X such that $\gamma\theta = \theta$.

3. The isometry γ is called elliptic if $l(\gamma) = 0$ and $l(\gamma)$ is achieved on X, in which case there exists a non empty convex subset F_γ of X such that, for any $x \in F_\gamma$, $\gamma x = x$.

The following result, due to G. Margulis, describes the structure of discrete subgroups of isometries generated by elements with small displacement.

Theorem 2.1 (G. Margulis, [5]) *There exists a constant $\mu(n,a) > 0$ such that if Γ is a discrete subgroup of the isometry group of (X,g), the subgroup Γ_μ of Γ generated by,*

$$S_\mu = \{\gamma \in \Gamma / \rho(x,\gamma x) \leq \mu(n,a)\},$$

is virtually nilpotent.

Given a set of isometries $S = \{\sigma_1, \dots, \sigma_p\}$ of (X, g), we define the "minimal displacement" of S by

Definition 2.1 $L(S) = \inf_{x \in X} \max_{i=1,\dots p} \rho(x, \sigma_i x)$

When Γ is a finitely generated discrete subgroup of the isometry group of (X, g), the above theorem 2.1 has the following

Corollary 2.1 *There exists a constant $\mu(n, a) > 0$ such that if Γ is a finitely generated not virtually nilpotent discrete subgroup of isometry of (X, g) and $S = \{\sigma_1, \dots, \sigma_p\}$ a finite generating set of Γ, then*

$$L(S) \geq \mu(n, a)\} \, .$$

In the following lemma we describe the structure of virtually nilpotent discrete subgroups of isometries of (X, g). Here by discrete we mean that the orbits are discrete sets in (X, g).

Lemma 2.1 *Let G be a discrete virtually nilpotent group of isometries of (X, g).*

a) If G contains an hyperbolic element γ, then G preserves the axis of γ.

b) If G contains a parabolic element γ with fixed point $\theta \in \partial X$, then G fixes the point θ.

c) If all elements of G are elliptic, then G is finite.

Proof. a) Let $\gamma \in G$ be an hyperbolic element and $\theta, \zeta \in \partial X$, the end points of the axis a_γ of γ. We claim that for any $\gamma' \in G$, then $\gamma'(\{\theta, \zeta\}) = \{\theta, \zeta\}$ or $\gamma'(\{\theta, \zeta\}) \cap \{\theta, \zeta\} = \emptyset$. Indeed assume that $\gamma'(\{\theta, \zeta\}) \cap \{\theta, \zeta\} = \{\theta\}$. The isometry $\gamma'\gamma\gamma'^{-1}$ is hyperbolic with axis $a_{\gamma'\gamma\gamma'^{-1}} = \gamma' a_\gamma$ equal to the geodesic joining θ and $\zeta' \neq \zeta$, where $\gamma'(\{\theta, \zeta\}) = \{\theta, \zeta'\}$. We may assume that θ is the attractive fixed point of γ and $\gamma'\gamma\gamma'^{-1}$ (replacing them by their inverse if necessary). Let $x \in a_\gamma$, then $(\gamma'\gamma\gamma'^{-1})^{-N}\gamma^N x$ is a sequence of pairwise distinct points which converges to a point on the axis $a_{\gamma'\gamma\gamma'^{-1}}$ of $\gamma'\gamma\gamma'^{-1}$. This contradicts the discreteness of G, proving thus the claim.

Now, if there exist $\gamma' \in G$ such that $\gamma'(\{\theta, \zeta\}) \cap \{\theta, \zeta\} = \emptyset$, the two hyperbolic isometries γ and $\gamma'\gamma\gamma'^{-1}$ would then have disjoint axis and therefore G would contain a free subgroup by a classical ping-pong argument. This would contradict the fact that G is virtually nilpotent. Consequently, for any $\gamma' \in G$, $\gamma'(\{\theta, \zeta\}) = \{\theta, \zeta\}$, which shows that G preserves the geodesic joining θ and ζ.

b) Let $\gamma \in G$ be a parabolic element, and $\theta \in \partial X$ its fixed point. If there exist $\gamma' \in G$ such that $\gamma'\theta \neq \theta$, then γ and $\gamma'\gamma\gamma'^{-1}$ would be to parabolic elements in G with distinct fixed point θ and $\gamma'\theta$ respectively. By a ping-pong argument, G would then contain a free subgroup, which contradicts the fact that G is virtually nilpotent. Thus G fixes $\theta \in \partial X$.

c) Let us now assume that all elements in G are elliptic. Let $N \subset G$ be a nilpotent subgroup of G with finite index. If $N = \{e\}$, then G is finite. We thus assume that $N \neq \{e\}$, the center $Z(N)$ of N is then not trivial. For

$g_1 \in Z(N)\setminus\{e\}$ let us denote $F_{g_1} \subset X$ the set of fixed points of g_1. Let $x_1 \in F_{g_1}$; by commutation of g_1 and $\exp_{x_1}$, we have $F_{g_1} = \exp_{x_1}(E_1)$, where E_1 is the eigenspace of $d_{x_1}g_1$ corresponding to the eigenvalue $+1$. This shows that F_{g_1} is a totally geodesic submanifold of X satisfying $\dim(F_{g_1}) < \dim(X)$, since $g_1 \neq e$. As every $\gamma \in N$ commutes with g_1, it satisfies $\gamma(F_{g_1}) = F_{g_1}$.

Let N_1 be the subgroup of $\mathrm{Isom}(F_{g_1})$ obtained by restriction to F_{g_1} of the elements of N; it is clearly nilpotent as the image of a nilpotent group. For $\gamma \in N$, the projection on F_{g_1} of any fixed point of γ is again a fixed point of γ; consequently, the elements of N_1 are elliptic elements of $\mathrm{Isom}(F_{g_1})$.

If $N_1 = \{e\}$, then F_{g_1} is pointwise fixed by N, therefore N is finite (the group is discrete and all elements have a common fixed point).

If $N_1 \neq \{e\}$, we may iterate the process. Indeed, let us suppose that we have constructed the totally geodesic submanifold F_{g_i}, we then construct N_i as the set of restrictions of elements of N to F_{g_i}, and, either $N_i = \{e\}$ in which case N is finite, or N_i is not trivial and, choosing $g_{i+1} \in Z(N_i) \setminus \{e\}$, we construct the totally geodesic submanifold $F_{g_{i+1}} \in F_{g_i}$ such that $\dim(F_{g_{i+1}}) < \dim(F_{g_i})$. This process stops for some $i_0 \leq n$ and then $N_{i_0} = \{e\}$ and $F_{g_{i_0}}$ is pointwise fixed by N and not empty. □

Lemma 2.2 *Let Γ be a finitely generated discrete group of isometries of (X, g).*
(i) If there exist a point $\theta \in \partial X$ fixed by Γ, then Γ is virtually nilpotent.
(ii) If Γ preserves a geodesic in X, then γ is virtually cyclic.

Proof. Proof of (i). There are three cases: 1) there is an hyperbolic element in Γ, 2) there is no hyperbolic element, but there is a parabolic element in Γ and 3) all elements in Γ are elliptic.

1) Let γ be a hyperbolic element in Γ, and a_γ its axis. One of the endpoints of a_γ is θ. As Γ is discrete, it follows from the argument in the proof of lemma 2.1 a) that for any $\gamma' \in \Gamma$, $\gamma'(\{\theta, \zeta\}) = \{\theta, \zeta\}$ or $\gamma'(\{\theta, \zeta\}) \cap \{\theta, \zeta\} = \emptyset$, where ζ is the other endpoint of a_γ. Therefore, $\gamma'(\{\theta, \zeta\}) = \{\theta, \zeta\}$ and $\gamma'(\theta) = \theta$ and $\gamma'(\zeta) = \zeta$. The group Γ preserves a_γ. Let us note that Γ does not contain any parabolic element, since such an element would fix θ and therefore also ζ which is impossible. The elements in Γ are thus either hyperbolic or elliptic.

Now, the projection on a_γ being distance decreasing, any element $\gamma' \in \Gamma$ achieves its displacement $l(\gamma')$ on the axis a_γ, and γ' is elliptic (resp. hyperbolic) iff $l(\gamma') = 0$ (resp. $l(\gamma') \neq 0$). Moreover, since $\gamma'(\theta) = \theta$, any elliptic element fixes pointwise the axis a_γ. The restriction to the axis a_γ is thus a morphism from Γ into the group of translations of the axis, whose kernel is the set of elliptic elements, which fix all points of a_γ and hence is finite. The group Γ is then virtually abelian.

2) In this case the elements of Γ are either elliptic or parabolic with fixed point θ. In particular, every element of Γ preserves each horospheres centred at θ. Indeed, this is clear for parabolic elements. Any elliptic element γ' fixes some point $x \in X$, and hence the whole geodesic c joining x to θ; let H be any horosphere centred at θ and y be its intersection with c, then γ' maps H onto the horosphere centred at $\gamma'(\theta) = \theta$ containing $\gamma'(y) = y$. This shows that $\gamma'(H) = H$.

Let $S = \{\sigma_1, \ldots, \sigma_p\}$ be a generating set of Γ, by the above discussion, $\inf_{x\in X} \max_{i\in\{1,\cdots,p\}} \rho(x, \gamma x) = 0$. In fact, for any geodesic c such that $c(+\infty) = \theta$, let H_t be the horosphere centred at θ and containing $c(t)$. The orthogonal projection from H_t to $H_{t+t'}$ contracts distances, we then get that $\rho(c(t), \gamma'(c(t))$ decreases to zero when t goes to infinity, for any $\gamma' \in \Gamma$. The group Γ is then virtually nilpotent by theorem 2.1.

3) If Γ only contains elliptic elements, then for any finite generating set $S = \{\sigma_1, \ldots, \sigma_p\}$, $\inf_{x\in X} \max_{i\in\{1,\cdots,p\}} \rho(x, \sigma_i x) = 0$, because each σ_i preserves each horosphere centred at θ, by the above argument. The group Γ is again virtually nilpotent.

Proof of (ii). A subgroup of index two of Γ fixes each endpoint of the globally preserved geodesic. Then we conclude as in the case 1 of (i). $\square$

For any two isometries $\gamma\,, \gamma'$ acting on (X, g) we define,

$$L(\gamma, \gamma') = \inf_{x\in X} \max\{\rho(x, \gamma x), \rho(x, \gamma' x)\}\,.$$

We now prove the,

Proposition 2.1 *Let Γ be a finitely generated discrete subgroup of* $\mathrm{Isom}(X, g)$*, where (X, g) is a Cartan-Hadamard manifold of sectional curvature $-a^2 \leq K_g \leq -1$. Let $S = \{\sigma_1, \ldots, \sigma_p\}$ be a finite generating set of Γ. If Γ is not virtually nilpotent, we have*

i) either there exist $\sigma_i, \sigma_j \in S$ such that the subgroup $< \sigma_i, \sigma_j >$ generated by these two elements is not virtually nilpotent and $L(\sigma_i, \sigma_j) \geq \mu(n, a)$,

ii) or all σ_i in S are elliptic and for all $\sigma_i \neq \sigma_j \in S$, either $< \sigma_i, \sigma_j >$ fixes some point in X and is finite, or it fixes a point $\theta \in \partial X$,

iii) or there exist $\sigma_i, \sigma_j, \sigma_k \in S$ such that $L(\sigma_i\sigma_j, \sigma_k) \geq \mu(n, a)$ and the group $< \sigma_i\sigma_j, \sigma_k >$ is not virtually nilpotent.

Proof. There are again three cases: a) there is a hyperbolic element in S, say σ_1; b) there is no hyperbolic element and there is a parabolic element in S, say σ_1; c) all σ_i's in S are elliptic.

a) Let us assume that σ_1 is hyperbolic. Let us consider all pairs (σ_1, σ_i) with $i = 2, \ldots, p$, and let us assume that $L(\sigma_1, \sigma_i) < \mu(n, a)$ for $i = 2, \ldots, p$. The groups $< \sigma_1, \sigma_i >$ are then virtually nilpotent. By lemma 2.1 a), every σ_i preserves the axis a_{σ_1} of σ_1, hence Γ preserves a_{σ_1} and is virtually nilpotent contradicting the assumption. Then there exist $\sigma_i \in S$ such that $L(\sigma_1, \sigma_i) \geq \mu(n, a)$ and $< \sigma_1, \sigma_i >$ is non virtually nilpotent.

b) Assume that σ_1 is parabolic with fixed point $\theta \in \partial X$. Let us consider all pairs (σ_1, σ_i), $i = 2, \ldots, p$, and assume that $< \sigma_1, \sigma_i >$ is virtually nilpotent (or that $L(\sigma_1, \sigma_i) < \mu(n, a)$), for all $i = 2, \ldots, p$). By lemma 2.1 b), σ_i fixes the point $\theta \in \partial X$, therefore Γ fixes θ and is virtually nilpotent, by lemma 2.2, a contradiction. Consequently, if σ_1 is parabolic, there exist $\sigma_i \neq \sigma_1$ such that $L(\sigma_1, \sigma_i) \geq \mu(n, a)$.

c) Let us assume that all σ_i's are elliptic, for $i = 2, \ldots, p$, and that for all pairs (σ_i, σ_j) the groups $< \sigma_i, \sigma_j >$ are virtually nilpotent (or that $L(\sigma_i, \sigma_j) < \mu(n,a)$). Let us denote $G = < \sigma_i, \sigma_j >$. There are again three cases: 1) there is a hyperbolic element in G, 2) there is no hyperbolic element and there is a parabolic element in G, 3) all elements in G are elliptic.

In the case 1), let γ be a hyperbolic element in G with axis a_γ. By lemma 2.1 a), G preserves a_γ. Since σ_i, σ_j are elliptic, they fix points x_i and x_j (respectively) on a_γ (recall that the displacement of σ_i and σ_j are achieved on a_γ by the distance decreasing property of the projection onto a_γ). If $x_i = x_j$, the G fixes x_i and it is thus finite. Let us now suppose that σ_i and σ_j do not fix the same point on a_γ, that is $x_i \neq x_j$ and none of the restriction $\tilde{\sigma}_i$ and $\tilde{\sigma}_j$ of σ_i and σ_j to a_γ is the identity. In that case, $\tilde{\sigma}_i$ and $\tilde{\sigma}_j$ are both symmetries around x_i and x_j, and then $\sigma_i\sigma_j$ is a hyperbolic element with axis a_γ. Let us then consider $< \sigma_i\sigma_j, \sigma_l >$ for $l = 1, \ldots, p$. Assume that for all $l = 1, \ldots, p$, $L(\sigma_i\sigma_j, \sigma_l) < \mu(n,a)$, the groups $< \sigma_i\sigma_j, \sigma_l >$ are then virtually nilpotent, and by lemma 2.1 a), all σ_l's preserve a_γ and hence Γ preserves a_γ and is thus virtually nilpotent which is a contradiction. Therefore, there exist $\sigma_k \in S$ such that $L(\sigma_i\sigma_j, \sigma_k) \geq \mu(n,a)$ and that $< \sigma_i\sigma_j, \sigma_k >$ is not virtually nilpotent.

In the case 2), let $\gamma \in G$ be a parabolic element with fixed point $\theta \in \partial X$. By lemma 2.1 b), G fixes θ.

In the case 3), all elements in G are elliptic and by lemma 2.1 c), G is finite. This ends the proof of the proposition. $\square$

3 Algebraic length and η-straight isometries

Let Γ be a finitely generated discrete group of isometries of (X, g) and $S = \{\sigma_1, \ldots, \sigma_p\}$ be a finite generating set of Γ.

Let us denote l_S and d_S the length and distance on the Cayley graph associated to S. Let x_0 be a point in X and define $L = \max_{i\in\{1,\ldots,p\}} \rho(x_0, \sigma_i x_0)$.

For any $\gamma \in \Gamma$ it follows from the triangle inequality that

$$\rho(x_0, \gamma x_0) \leq l_S(\gamma) L \,. \tag{3}$$

Let η be a positive number such that $0 < \eta < L$.

Definition 3.1 *An isometry γ of Γ is said to be (L,η)-straight if $\rho(x_0, \gamma x_0) \geq (L - \eta) l_S(\gamma)$.*

Remark. Notice that the above definition depends on the choice of x_0 and of a generating set S.

When Γ is a finitely generated discrete group, for any finite generating set $S = \{\sigma_1, \ldots, \sigma_p\}$ we define,

$$L(S) = \inf_{x\in X} \max_{i\in\{1,\ldots,p\}} \rho(x, \sigma_i x) \,.$$

When Γ is not virtually nilpotent, by theorem 2.1, for any finite generating set S, $L = L(S) \geq \mu(n,a) > 0$, where $\mu(n,a)$ is the Margulis constant. We then have,

Lemma 3.1 *Let Γ be a finitely generated non virtually nilpotent discrete group of isometries of (X, g). For any finite generating set $S = \{\sigma_1, \dots, \sigma_p\}$ of Γ, there exist $x_0 \in X$ such that,*

$$L(S) = \inf_{x\in X} \max_{i\in\{1,\dots,p\}} \rho(x, \sigma_i x) = \max_{i\in\{1,\dots,p\}} \rho(x_0, \sigma_i x_0)\,.$$

Proof. Let us assume that the infimum in the definition of $L(S)$ is not achieved in X, then there exist a sequence of points $x_k \in X$, which satisfies $\max_{i\in\{1,\dots,p\}} \rho(x_k, \sigma_i x_k) \to L(S)$ when $k \to \infty$, and x_k converges to a point, say θ, in ∂X. For k large enough and $i \in \{1, \dots, p\}$, we then have $\rho(x_k, \sigma_i x_k) \le L+1$ and hence $\sigma_i\theta = \theta$ for all i. This shows that Γ fixes θ and is thus virtually nilpotent by lemma 2.2, which contradicts the hypothesis. □

In the sequel of this section, we shall show that if G is a finitely generated discrete group of isometries of (X, g), for any finite generating set $S = \{\sigma_1, \dots, \sigma_p\}$ of G such that each σ_i has a displacement $l(\sigma_i)$ small compared to $L(S)$, then there exist many non-$(L(S), \eta)$-straight elements in G for a constant η to be defined.

We need the following geometric lemmas.

Lemma 3.2 *Let (x_1, x_2, x_3) be a geodesic triangle in (X, g), where (X, g) is a Cartan-Hadamard manifold with $K_g \le -1$. Let x_2' be the point in the segment $[x_1, x_3]$ dividing it in two segments of length proportional to $L_1 := \rho(x_1, x_2)$ and $L_2 := \rho(x_2, x_3)$. We have,*

$$\rho(x_2', x_2) \le \operatorname{Argcosh}\Big[\exp\Big(\alpha\big(\rho(x_1, x_2) + \rho(x_2, x_3) - \rho(x_1, x_3)\big)\Big)\Big]\,,$$

where $\alpha = \frac{\max(L_1, L_2)}{L_1+L_2}$.

Proof. We consider a comparison geodesic triangle (y_1, y_2, y_3) in the Poincaré disk $(\mathbb{H}^2, d)$ of constant curvature -1 such that $d(y_i, y_j) = \rho(x_i, x_j)$ for all $i, j \in \{1, 2, 3\}$. Let y_2' be the point of the segment $[y_1, y_3]$ dividing it in two segments of length proportional to L_1 and L_2. Since (X, g) is a $CAT(-1)$ space we have

$$\rho(x_2, x_2') \le d(y_2, y_2'). \tag{4}$$

One of the two triangles (y_1, y_2', y_2), (y_3, y_2', y_2) has angle at y_2' greater than or equal to $\pi/2$, therefore from hyperbolic trigonometry formulae we get the existence of $i \in \{1, 2\}$ such that

$$\cosh L_i \ge \cosh\big[d(y_2, y_2')\big] \cosh\Big[\frac{L_i}{(L_1 + L_2)} d(y_1, y_3)\Big] \tag{5}$$

Let us denote $\Delta = \rho(x_1, x_2) + \rho(x_2, x_3) - \rho(x_1, x_3)$. We have

$$\frac{L_i}{L_1 + L_2} d(y_1, y_3) \ge L_i - \alpha\Delta, \tag{6}$$

where $\alpha = \frac{\max(L_1,L_2)}{L_1+L_2}$. Therefore from (4) and (5) we get

$$\cosh\left[\rho(x_2', x_2)\right] \leq \frac{\cosh L_i}{\cosh(L_i - \alpha\Delta)}\,, \tag{7}$$

hence

$$\cosh\left[\rho(x_2', x_2)\right] \leq e^{\alpha\Delta}. \tag{8}$$

□

Lemma 3.3 *Let (X, g) be a Cartan-Hadamard manifold with sectional curvature $K_g \leq -1$. Let δ, L be any positive numbers such that $L >$ Argcosh(e^δ). Then, for any isometry γ of (X, g) such that its displacement $l(\gamma)$ satisfies $l(\gamma) \leq \delta$, and for any point $x_0 \in X$ such that $\rho(x_0, \gamma x_0) \geq L$, we have*

$$\rho(x_0, \gamma^2 x_0) \leq 2\rho(x_0, \gamma x_0) - \left(1 - \frac{e^\delta}{\cosh L}\right)^2,$$

Proof. Let us consider $\Delta = 2\rho(x_0, \gamma x_0) - \rho(x_0, \gamma^2 x_0)$. We want to prove that $\Delta \geq \left(1 - \frac{e^\delta}{\cosh L}\right)^2$. By assumption there is a point $y \in X$ such that $\rho(y, \gamma y) \leq \delta$. Let us write $L_1 =: \rho(x_0, \gamma y)$, $L_2 =: \rho(\gamma^2 x_0, \gamma y)$ and $L' =: \rho(x_0, y)$. By the triangle inequality we have for $i = 1, 2$

$$L' - \delta \leq L_i \leq L' + \delta\,. \tag{9}$$

Let us associate to the triangle $(x_0, \gamma y, \gamma^2 x_0)$ the comparison triangle (z_1, z_2, z_3) in the hyperbolic plane $(\mathbb{H}^2, d)$ such that $d(z_1, z_2) = L_1$, $d(z_2, z_3) = L_2$ and $d(z_1, z_3) = \rho(x_0, \gamma^2 x_0)$. Let x [resp. z] be the middle point of the segment $(x_0, \gamma^2 x_0)$ [resp. (z_1, z_3)]. One of the two triangles (z_2, z, z_1) or (z_2, z, z_3) has angle at z greater than or equal to $\pi/2$. Let us assume without restriction that this triangle is (z_2, z, z_1), then the hyperbolic trigonometric formulas give

$$\cosh L_1 \geq \cosh\Big[d(z_2, z)\Big] \cosh\left[\frac{1}{2} d(z_1, z_3)\right]$$

therefore from (9) we get

$$\cosh(L' + \delta) \geq \cosh\Big[d(z_2, z)\Big] \cosh\left[\frac{1}{2} d(z_1, z_3)\right]$$

and since (X, g) is a $CAT(-1)$ space we have $\rho(x, \gamma y) \leq d(z_2, z)$, thus we obtain

$$\cosh(L' + \delta) \geq \cosh\Big[\rho(x, \gamma y)\Big] \cosh\left[\frac{1}{2}\rho(x_0, \gamma^2 x_0)\right]. \tag{10}$$

Let us write $L_0 = \rho(x_0, \gamma x_0)$. By the triangle inequality we have

$$\rho(x, \gamma y) \geq |\rho(\gamma y, \gamma x_0) - \rho(\gamma x_0, x)|\,,$$

therefore, since $\rho(\gamma y, \gamma x_0) = \rho(y, x_0) = L'$ and $\frac{1}{2}\rho(\gamma^2 x_0, x_0) = L_0 - \frac{\Delta}{2}$, we get from 10

$$\cosh(L' + \delta) \geq \cosh\Big(L' - \rho(\gamma x_0, x)\Big) \cosh\Big(L_0 - \frac{\Delta}{2}\Big). \tag{11}$$

We get from (11),

$$(\cosh\delta + \sinh\delta)\cosh L' \geq \Big(\cosh\big[\rho(\gamma x_0, x)\big] - \sinh\big[\rho(\gamma x_0, x)\big]\Big)\big(\cosh L'\big)\cosh\big(L_0 - \frac{\Delta}{2}\big)$$

hence

$$e^{\delta} \geq \Big(\cosh\big[\rho(\gamma x_0, x)\big] - \sinh\big[\rho(\gamma x_0, x)\big]\Big)\cosh\big(L_0 - \frac{\Delta}{2}\big). \tag{12}$$

Now applying the inequality 7 in the proof of lemma 3.2 we have

$$\cosh\big[\rho(\gamma x_0, x)\big] \leq \frac{\cosh L_0}{\cosh\big(L_0 - \frac{\Delta}{2}\big)},$$

and since $\cosh r - \sinh r = e^{-r}$ is a decreasing function of r we get from (12)

$$e^{\delta} \geq \cosh L_0 - \Big(\cosh^2 L_0 - \cosh^2\big(L_0 - \frac{\Delta}{2}\big)\Big)^{\frac{1}{2}}. \tag{13}$$

But we can check that $\cosh^2 L_0 - \cosh^2\big(L_0 - \frac{\Delta}{2}\big)\Big) \leq \Delta \cosh^2 L_0$ so we get from (13)

$$e^{\delta} \geq \cosh(L_0)\big(1 - \Delta^{\frac{1}{2}}\big)$$

and therefore

$$\Delta \geq \Big(1 - \frac{e^{\delta}}{\cosh L_0}\Big)^2,$$

when $e^{\delta} < \cosh L$. The lemma now follows whenever $L_0 \geq L$.

$\square$

Lemma 3.4 *Let (X, g) be a Cartan-Hadamard manifold with sectional curvature $K_g \leq -1$. Let us consider four points y_0, y_1, y_2, y_3 such that*

$$\rho(y_0, y_1) + \rho(y_1, y_2) - \rho(y_0, y_2) \leq \eta_1$$

and

$$\rho(y_1, y_2) + \rho(y_2, y_3) - \rho(y_1, y_3) \leq \eta_2$$

then

$$\rho(y_0, y_1) + \rho(y_1, y_2) + \rho(y_2, y_3) - \rho(y_0, y_3) \leq \Big(1 + \frac{\rho(y_2, y_3)}{\rho(y_1, y_2)}\Big)\Big(\eta_1 + \operatorname{Argcosh} e^{\eta_2}\Big).$$

Proof. For $i = 1, 2, 3$ let us write $L_i = \rho(y_{i-1}, y_i)$. Let y_2' be the point on the segment (y_1, y_3) dividing it in two segments of length proportional to L_2 and L_3. By lemma 3.2 we have

$$\rho(y_2, y_2') \leq \text{Argcosh}\left(e^{\eta_2}\right). \tag{14}$$

Since $\rho(y_0, y_1) + \rho(y_1, y_2) - \rho(y_0, y_2) \leq \eta_1$ by assumption we get from (14) and the triangle inequality

$$\rho(y_0, y_2') \geq \rho(y_0, y_2) - \rho(y_2, y_2') \geq \rho(y_0, y_1) + \rho(y_1, y_2) - \left[\eta_1 + \text{Argcosh}\left(e^{\eta_2}\right)\right] \tag{15}$$

On the other hand by convexity of the distance function on (X, g) we get

$$\rho(y_0, y_2') \leq \frac{L_3}{L_2 + L_3}\rho(y_0, y_1) + \frac{L_2}{L_2 + L_3}\rho(y_0, y_3) \tag{16}$$

The inequalities (15) and (16) give

$$\rho(y_0, y_3) \geq \rho(y_0, y_1) + L_2 + L_3 - \left(\frac{L_2 + L_3}{L_2}\right)\left(\eta_1 + \text{Argcosh}\left(e^{\eta_2}\right)\right)$$

and the lemma follows. □

Lemma 3.5 *Let L and η be two positive numbers such that,*

$$\eta < \min\left(\frac{L}{4}, \frac{1}{2}\log\left[\frac{1}{2}\left(\cosh(\frac{L}{2}) + \frac{1}{\cosh(\frac{L}{2})}\right)\right]\right).$$

Let (X, g) be a Cartan-Hadamard manifold with sectional curvature $K_g \leq -1$. We consider two elliptic isometries γ_1, γ_2 of (X, g) with a common fixed point $y \in X \cup \partial X$. If we assume that $L - \eta \leq \rho(x_0, \gamma_1 x_0) \leq L$ and that $L - \eta \leq \rho(x_0, \gamma_2 x_0) \leq L$, then

$$\rho(x_0, \gamma_1\gamma_2 x_0) < 2(L - \eta).$$

Proof. We first claim that in both cases, $y \in X$ and $y \in \partial X$, there exist some sequence $(u_k)_{k\in\mathbf{N}}$ of points in X converging to y such that $\rho(u_k, \gamma_1\gamma_2 x_0) = \rho(u_k, x_0) = l_k$ and that the quantity $\epsilon_k = |\rho(u_k, \gamma_1 x_0) - l_k|$ goes to zero when k goes to $+\infty$; in fact, when γ_1 and γ_2 fix some point $y \in X$ we may choose $u_k = y$ for every k. If γ_1 and γ_2 fix $y \in \partial X$, they also preserve each horosphere centred at y (see the proof of lemma 2.2 2)), and thus x_0, $\gamma_1 x_0$ and $\gamma_1\gamma_2 x_0$ lie on the same horosphere centred at y. Approximating this horosphere by a sequence $(S_k)_{k\in\mathbf{N}}$ of spheres passing through x_0 and $\gamma_1\gamma_2 x_0$ and denoting u_k the centre of S_k, we get that $\rho(u_k, \gamma_1 x_0) - \rho(O, u_k)$ and $\rho(u_k, x_0) - \rho(O, u_k)$ simultaneously go to $B(\gamma_1 x_0, y) = B(x_0, y)$ (where O is some fixed origin in X and B the Busemann function normalised at O). This proves the claim.

Consider the triangle $(u_k, v, w) = (u_k, x_0, \gamma_1\gamma_2 x_0)$ and z the point of the geodesic segment $[v, w]$ which divides it in two segments of length proportional to $L_1 =: \rho(v, \gamma_1 x_0)$ and $L_2 =: \rho(w, \gamma_1 x_0)$. Let us recall that by assumption we have $L - \eta \leq L_i \leq L$.

We consider the comparison triangle $(\bar{u}_k, \bar{v}, \bar{w})$ on the two-dimensional hyperbolic space $\mathbf{H}^2$ such that $d(\bar{u}_k, \bar{v}) = \rho(u_k, v) = l_k = \rho(u_k, w) = d(\bar{u}_k, \bar{w})$ and $d(\bar{v}, \bar{w}) = \rho(v, w)$, where d is the hyperbolic distance on $\mathbf{H}^2$. Let $\bar{z}$ be the point of the segment $[\bar{u}, \bar{v}]$ dividing it in two segments of length proportional to L_1 and L_2. Let us write $L_1' = \rho(v, z)$ and $L_2' = \rho(w, z)$. We now consider the triangle $(\bar{u}_k, \bar{v}, \bar{z})$ or $(\bar{u}_k, \bar{w}, \bar{z})$, namely the one which has angle at $\bar{z}$ larger than or equal to $\pi/2$. We can assume without restriction that this triangle is $(\bar{u}_k, \bar{v}, \bar{z})$. The hyperbolic trigonometry formulas then show that the point $\bar{z}$ satisfies,

$$\cosh(l_k) \geq \cosh(L_1') \cosh(d(\bar{u}_k, \bar{z})) .$$

Since (X, g) is a CAT(-1)-space, we get that,

$$\rho(u_k, z) \leq d(\bar{u}_k, \bar{z}) ,$$

and thus that,

$$\cosh(\rho(u_k, z)) \leq \frac{\cosh(l_k)}{\cosh(L_1')} . \qquad (17)$$

On the other hand, the triangle inequality implies that $\rho(u_k, z) \geq l_k - \epsilon_k - \rho(\gamma_1 x_0, z)$ and thus that

$$\cosh(\rho(u_k, z)) \geq e^{-(\rho(\gamma_1 x_0, z)+\epsilon_k)} \cosh(l_k) .$$

Plugging this in formula 17 and letting $\epsilon_k \to 0$, we get:

$$e^{\rho(\gamma_1 x_0, z)} \geq \cosh(L_1') . \qquad (18)$$

On the other hand, by lemma 3.2, we have

$$\cosh(\rho(\gamma_1 x_0, z)) \leq \exp\left(\max\Big\{\rho(v, \gamma_1 x_0), \rho(w, \gamma_1 x_0)\Big\} \Big(1 - \frac{\rho(v, w)}{\rho(v, \gamma_1 x_0) + \rho(w, \gamma_1 x_0)}\Big) \right) .$$

and hence

$$\cosh(\rho(\gamma_1 x_0, z)) \leq e^{(L - \frac{\rho(v,w)}{2})} . \qquad (19)$$

Let us now assume, by contradiction, that

$$\rho(v, w) = \rho(x_0, \gamma_1 \gamma_2 x_0) > 2(L - \eta) .$$

Plugging this in the inequalities (18) and (19) we obtain, using the fact that $x \to x + 1/x$ is an increasing function for $x > 1$:

$$\cosh(L_1') + \frac{1}{\cosh(L_1')} \leq 2\cosh(\rho(\gamma_1 x_0, z)) \leq 2e^{\eta}. \qquad (20)$$

Now since $\frac{L_1'}{L_2'} = \frac{L_1}{L_2}$, we also obtain

$$L_1' = (L_1' + L_2')\big(\frac{L_1}{L_1 + L_2}\big) \geq \frac{2(L - \eta)(L - \eta)}{2L} \geq L - 2\eta$$

which gives by inequalities (20)

$$\cosh(L-2\eta)+\frac{1}{\cosh(L-2\eta)}\leq 2\cosh(\rho(\gamma_1 x_0,z))\leq 2e^{\eta}. \tag{21}$$

we then get a contradiction when

$$\eta<\min\Big(\frac{L}{4},\frac{1}{2}\log\Big[\frac{1}{2}\big(\cosh(\frac{L}{2})+\frac{1}{\cosh(\frac{L}{2})}\big)\Big]\Big)\,.$$

$\square$

Let Γ be a finitely generated discrete group of isometries of (X,g) and $S=\{\sigma_1,\dots,\sigma_p\}$ be a finite generating set. Let us assume that Γ is not virtually nilpotent and recall that $L(S)=\inf_{x\in X}\max_{i\in\{1,\dots,p\}}\rho(x,\sigma_i x)$. By lemma 3.1 we have $L(S)=\max_{i\in\{1,\dots,p\}}\rho(x_0,\sigma_i x_0)$, for some point $x_0\in X$, and by corollary 2.1, $L(S)\geq\mu(n,a)>0$. Let us recall that for $0\leq\eta\leq L$, an element $\gamma\in\Gamma$ is said to be (L,η)-straight if

$$\rho(x_0,\gamma\,x_0)>(L-\eta)l_S(\gamma)\,.$$

In the following two propositions we give conditions under which there are many non (L,η)-straight elements in Γ.

Proposition 3.1 *Let (X,g) be a Cartan Hadamard manifold whose sectional curvature satisfies $-a^2\leq K\leq -1$ and Γ a discrete non virtually nilpotent group of isometries of (X,g) generated by $S=\{\sigma_1,\dots,\sigma_p\}$. Let us assume that all σ_i's are elliptic and that for all $\sigma_i\neq\sigma_j\in S$, the group $<\sigma_i,\sigma_j>$ fixes a point $y\in X$ or $\theta\in\partial X$. Let η be a positive number such that*

$$\eta<\min\left(\frac{L}{4},\frac{1}{2}\Big(1-\frac{1}{\cosh(L)}\Big)^2,\frac{1}{2}\log\Big[\frac{1}{2}(\cosh\frac{L}{2}+\frac{1}{\cosh\frac{L}{2}})\Big]\right),$$

where $L=L(S)=\inf_{x\in X}\max_{i\in\{1,\dots,p\}}\rho(x,\sigma_i x)=\max_{i\in\{1,\dots,p\}}\rho(x_0,\sigma_i x_0)$, then any $\gamma\in\Gamma$ with $l_S(\gamma)=2$, i.e $\gamma=\sigma_i^2$ or $\gamma=\sigma_i\sigma_j$, is not $(L,\eta/2)$-straight, that is, $\rho(x_0,\gamma x_0)\leq 2(L-\eta/2)$.

Proof. Consider the case where $\gamma=\sigma_i^2$. If σ_i is not (L,η)-straight, we have, by the triangle inequality, $\rho(x_0,\sigma_i^2 x_0)\leq 2(L-\eta)$. If σ_i is (L,η)-straight, we have by lemma 3.3,

$$\rho(x_0,\sigma_i^2 x_0)\leq 2L-\left(1-\frac{1}{\cosh L}\right)^2\leq 2(L-\eta)\,.$$

Let us now consider the case where $\gamma=\sigma_i\sigma_j$, for $i\neq j$. If σ_i or σ_j is not (L,η)-straight, we have, by the triangle inequality,

$$\rho(x_0,\sigma_i\sigma_j x_0)\leq\rho(x_0,\sigma_i x_0)+\rho(x_0,\sigma_j x_0)\leq L+(L-\eta)\,,$$

therefore, $\rho(x_0,\sigma_i\sigma_j x_0)\leq 2(L-\eta/2)$.

If σ_i and σ_j are (L,η)-straight, lemma 3.5 implies that $\rho(x_0, \sigma_i\sigma_j x_0) \leq 2(L-\eta)$. $\square$

In the next proposition we will assume that all elements $\gamma \in \Gamma$ whose algebraic length is less than or equal to 4 have a displacement smaller than δ where

$$\delta = \log\left[\cosh(\frac{L}{4})\right], \tag{22}$$

and we set

$$\eta = 10^{-3}\Big(1 - \frac{\cosh(\frac{L}{4})}{\cosh(\frac{L}{2})}\Big)^4. \tag{23}$$

We will find in that case many non (L,η)-straight elements.

Proposition 3.2 *Let (X,g) be a Cartan-Hadamard manifold whose sectional curvature satisfies $a^2 \leq K_g \leq -1$ and G a discrete non virtually nilpotent group of isometries of (X,g) generated by a set of two isometries $\Sigma = \{\sigma_1, \sigma_2\}$. Let $L = \inf_{x\in X} \max\{\rho(x,\sigma_1 x), \rho(x,\sigma_2 x)\}$, and $\Sigma = \{\sigma_1,\sigma_2\}$. Let η and δ be the numbers defined in (23) and (22). We assume that $l(\gamma') < \delta$ for all $\gamma' \in G$ such that $l_\Sigma(\gamma') < 4$. Then all elements $\gamma \in G$ such that $l_\Sigma(\gamma) = 6$ are not (L,η)-straight.*

We will need the following lemmas.

Lemma 3.6 *Let $\gamma = a\gamma' b \in G$ be such that $l_\Sigma(\gamma) = l_\Sigma(a) + l_\Sigma(\gamma') + l_\Sigma(b)$. If γ is (L,η)-straight, then γ' is $(L, C\,\eta)$-straight where $C = \frac{l_\Sigma(\gamma)}{l_\Sigma(\gamma')}$.*

Proof. Let us note that by definition of $L = L(\Sigma)$, we have for any $\gamma \in G$

$$\rho(x_0, \gamma x_0) \leq L\,.l_\Sigma(\gamma)\,.$$

By triangle inequality we have

$$\rho(x_0,\gamma x_0) \leq \rho(x_0, a x_0) + \rho(x_0, \gamma' x_0) + \rho(x_0, b x_0)\,,$$

hence by assumption on γ we get

$$(L-\eta)\, l_\Sigma(a\gamma' b) \leq L\,\big(l_\Sigma(a) + l_\Sigma(b)\big) + \rho(x_0, \gamma' x_0)$$

and therefore,

$$\rho(x_0, \gamma' x_0) \geq L\, l_\Sigma(\gamma') - \eta\, l_\Sigma(\gamma) \geq (L - C\,\eta) l_\Sigma(\gamma')\,.$$

$\square$

Lemma 3.7 *Let α, β be two elements of G distinct of the neutral element and such that $l_\Sigma(\alpha) \leq 2$, $l_\Sigma(\beta) \leq 2$. Under the assumptions of the proposition 3.2, if γ is (L,η)-straight with $l_\Sigma(\gamma) = 6$, then any reduced word representing γ does not contain (i) α^2 or (ii) $\alpha\beta\alpha$.*

Assuming the lemma 3.7, the proof of the proposition 3.2 can be finished as follows:

Proof. Let $\gamma \in G$ of length $l_\Sigma(\gamma) = 6$. Let us write γ as a reduced word in the generators of Σ, $\gamma = \sigma_{i_1}^{p_1} . \dots . \sigma_{i_k}^{p_k}$, where $\sigma_{i_j} = \sigma_1$ or $\sigma_{i_j} = \sigma_2$, $p_j \in \mathbb{Z}^*$, $i_j \neq i_{j+1}$ and $i_j = i_{j+2}$. Arguing by contradiction we assume that γ is η-straight. Then, by lemma 3.7 (i), all p_j are equal to $+1$ or -1 and in particular we have $k = 6$. Therefore $\gamma = \sigma_{i_1}^{p_1} . \sigma_{i_2}^{p_2} . \sigma_{i_3}^{p_3} . \sigma_{i_4}^{p_4} . \sigma_{i_5}^{p_5} . \sigma_{i_6}^{p_6}$. By lemma 3.7 (ii) we also have $p_{j+2} \neq p_j$ hence $p_{j+2} = -p_j$ so $\gamma = \sigma_{i_1}^{p_1} . \sigma_{i_2}^{p_2} . \sigma_{i_1}^{-p_1} . \sigma_{i_2}^{-p_2} . \sigma_{i_1}^{p_1} . \sigma_{i_2}^{p_2}$, which is impossible by lemma 3.7 (ii) with $\alpha = \sigma_{i_1}^{p_1} . \sigma_{i_2}^{p_2}$ and $\beta = \sigma_{i_1}^{-p_1} . \sigma_{i_2}^{-p_2}$. This finishes the proof of proposition 3.2. ☐

Let us now prove the lemma 3.7:

Proof. We first claim that if L, η and δ are chosen as in the proposition 3.2 then we have

$$\eta \leq \frac{L}{4000} \tag{24}$$

and

$$12\eta + \text{Argcosh}\left(e^{12\eta}\right) \leq \frac{1}{4}\Big(1 - \frac{e^\delta}{\cosh(L/2)}\Big) . \tag{25}$$

Proof of the claim. By definition of η, (cf. (23)), we have

$$1000\eta = \Big(1 - \frac{\cosh(L/4)}{\cosh(L/2)}\Big)^4$$

therefore

$$1000\eta < \frac{\cosh(L/2) - \cosh(L/4)}{\cosh(L/2)}$$

and

$$1000\eta < \frac{\sinh(L/2).L/4}{\cosh(L/2)} < \frac{L}{4},$$

which proves the first inequality of the claim. On the other hand, let $x \in]0, 1[$, then $e^x \leq 1+2x \leq \cosh(2\sqrt{x})$. Choosing $x = 12\eta$ we obtain, using the inequality $\eta < \frac{1}{1000}$, that

$$12\eta + \text{Argcosh}(e^{12\eta}) \leq 12\eta + 2\sqrt{12\eta} < \frac{1}{4}\sqrt{1000\eta}$$

therefore we get

$$12\eta + \text{Argcosh}(e^{12\eta}) \leq \frac{1}{4}\Big(1 - \frac{\cosh(L/4)}{\cosh(L/2}\Big)^2 \leq \frac{1}{4}\Big(1 - \frac{e^\delta}{\cosh(L/2)}\Big),$$

which ends the proof of the claim.

Proof of lemma 3.7 (i). Let us assume that $\gamma = a\alpha^2 b$ is (L, η)-straight, and $l_\Sigma(\gamma) = 6$. Then by lemma 3.6 α and α^2 are $(L, 3\eta)$-straight. We then get with (24)

$$\rho(x_0, \alpha x_0) > (L - 3\eta) l_\Sigma(\alpha) > \frac{L}{2} .$$

On the other hand since $l(\alpha) \leq \delta$ and Argcosh$[e^\delta] = \frac{L}{4} < \frac{L}{2}$, we can apply lemma 3.3 to α replacing L by $L/2$ and get

$$\rho(x_0, \alpha^2 x_0) < 2\rho(x_0, \alpha x_0) - \Big(1 - \frac{e^\delta}{\cosh{(L/2)}}\Big)^2 .$$

We then get by the choice of η, cf. (23),

$$\rho(x_0, \alpha^2 x_0) < (L - 3\eta)\, l_\Sigma(\alpha^2)$$

which contradicts the fact that α^2 is $(L, 3\eta)$-straight and concludes the proof of lemma 3.7 (i).

Proof of lemma 3.7 (ii). Let us assume that $\gamma = a\alpha\beta\alpha b$ is (L, η)-straight, and $l_\Sigma(\gamma) = 6$. The Lemma 3.6 says that $\alpha\beta\alpha$ is $(L, 2\eta)$-straight and that $\alpha\beta$ is $(L, C'\,\eta)$-straight where $C' = 2\frac{l_\Sigma(\alpha\beta\alpha)}{l_\Sigma(\alpha\beta)}$. Since $\alpha\beta\alpha$ is $(L, 2\eta)$-straight, we have by triangle inequality

$$(L - 2\eta) l_\Sigma(\alpha\beta\alpha) \leq 2\rho(x_0, \alpha\, x_0) + L\, l_\Sigma(\beta)$$

and therefore

$$2\rho(x_0, \alpha\, x_0) \geq (L - 2\eta) l_\Sigma(\alpha\beta\alpha) - L\, l_\Sigma(\beta) = L\, l_\Sigma(\alpha^2) - 2\eta\, l_\Sigma(\alpha\beta\alpha)$$

hence we obtain

$$\rho(x_0, \alpha\, x_0) \geq L\, l_\Sigma(\alpha) - \eta\, l_\Sigma(\alpha\beta\alpha)\,,$$

and since $l_\Sigma(\alpha) \leq 2$ and $l_\Sigma(\beta) \leq 2$, we deduce that

$$\rho(x_0, \alpha\, x_0) \geq (L - 4\eta)\, l_\Sigma(\alpha)\,,$$

that is α is $(L, 4\eta)$-straight. We set $x_1 = \alpha\beta\, x_0$, $x_2 = \alpha\beta\alpha\, x_0$ and $x_3 = (\alpha\beta)^2\, x_0 = \alpha\beta\alpha\beta\, x_0$. We get, since $\alpha\beta\alpha$ is $(L, 2\eta)$-straight,

$$\begin{aligned}
\rho(x_0, x_1) + \rho(x_1, x_2) - \rho(x_0, x_2) &= \rho(x_0, \alpha\beta\, x_0) + \rho(x_0, \alpha\, x_0) - \rho(x_0, \alpha\beta\alpha\, x_0) \\
&\leq L\big[l_\Sigma(\alpha\beta) + l_\Sigma(\alpha)\big] - (L - 2\eta) l_\Sigma(\alpha\beta\alpha) \\
&\leq 12\eta\,.
\end{aligned}$$

In the same way, since $\alpha\beta$ is $(L, C'\,\eta)$-straight with $C' = 2\frac{l_\Sigma(\alpha\beta\alpha)}{l_\Sigma(\alpha\beta)}$, we have

$$\begin{aligned}
\rho(x_1, x_2) + \rho(x_2, x_3) - \rho(x_1, x_3) &= \rho(x_0, \alpha\, x_0) + \rho(x_0, \beta\, x_0) - \rho(x_0, \alpha\beta\, x_0) \\
&\leq L\big[l_\Sigma(\alpha) + l_\Sigma(\beta)\big] - (L - C'\eta) l_\Sigma(\alpha\beta) \\
&\leq 2\eta l_\Sigma(\alpha\beta\alpha) \\
&\leq 12\eta\,.
\end{aligned}$$

We can therefore apply the lemma 3.4 and get

$$\begin{aligned}
2\rho(x_0, \alpha\beta\, x_0) - \rho\big(x_0, (\alpha\beta)^2\, x_0\big) &\leq \rho(x_0, \alpha\beta\, x_0) + \rho(x_0, \alpha\, x_0) + \rho(x_0, \beta\, x_0) - \rho\big(x_0, (\alpha\beta)^2\, x_0\big) \\
&= \rho(x_0, x_1) + \rho(x_1, x_2) + \rho(x_2, x_3) - \rho(x_0, x_3) \\
&\leq \Big(1 + \frac{\rho(x_0, \beta\, x_0)}{\rho(x_0, \alpha\, x_0)}\Big)\big(12\eta + Argcosh[e^{12\eta}]\big) \\
&\leq \Big(1 + \frac{L\, l_\Sigma(\beta)}{(L - 4\eta) l_\Sigma(\alpha)}\Big) . \frac{1}{4}\Big(1 - \frac{e^\delta}{\cosh(L/2)}\Big)^2 ,
\end{aligned}$$

the last inequality coming from (25) and the fact that α is $(L, 4\eta)$-straight. From (24) we therefore get

$$\rho\big(x_0, (\alpha\beta)^2\, x_0\big) \geq 2\rho(x_0, \alpha\beta\, x_0) - \Big(1 - \frac{e^{\delta}}{\cosh(L/2)}\Big)^2 . \tag{26}$$

On the other hand we have seen that $\alpha\beta$ is $(L, C'\,\eta)$-straight with $C' = 2\frac{l_\Sigma(\alpha\beta\alpha)}{l_\Sigma(\alpha\beta)}$, so that

$$\rho(x_0, \alpha\beta\, x_0) \geq (L - C'\eta)\, l_\Sigma(\alpha\beta) \geq 2L - 2\eta\, l_\Sigma(\alpha\beta\alpha)\,,$$

and since $l_\Sigma(\alpha\beta\alpha) \leq 6$ the above inequality gives with (24)

$$\rho(x_0, \alpha\beta\, x_0) \geq L\,. \tag{27}$$

By assumption, since $l_\Sigma(\alpha\beta) \leq 4$, the displacement of $\alpha\beta$ satisfies $l(\alpha\beta) \leq \delta$, and with (27) we can apply the lemma 3.3 to get

$$\rho\big(x_0, (\alpha\beta)^2 x_0\big) \leq 2\rho(x_0, \alpha\beta\, x_0) - \Big(1 - \frac{e^{\delta}}{\cosh(L/2)}\Big)^2$$

which contradicts (26). This concludes the proof of the lemma 3.7 and the proposition 3.2. $\square$

4 Mapping the Cayley graph of G into X.

Let G be a finitely generated discrete group of isometries of (X, g) a Cartan Hadamard manifold of sectional curvature $-a^2 \leq K \leq -1$. We consider S a finite generating set of G and the Cayley graph $\mathcal{G}_S$ of G associated to S. We define a distance d_S on $\mathcal{G}_S$ in the following way: each edge is isometric to the segment $[0, 1] \subset \mathbb{R}$ and the distance $d_S(\gamma, \gamma')$ between two vertices γ, γ' of $\mathcal{G}_S$ is the word distance $d_S(\gamma, \gamma') = l_S(\gamma^{-1}\gamma')$. The group G acts by isometries on $(\mathcal{G}_S, d_S)$ and on (X, g). The goal of this section is to construct for each number c large enough an equivariant map $f_c : \mathcal{G}_S \to X$ such that f_c is Lipschitzian of Lipschitz constant c.

4.1 Poincaré series, measures and convexity.

We first consider the Poincaré series,

$$P_c(s, x, y) = \Sigma_{\gamma\in G} e^{-c d_S(s,\gamma)} \cosh\big[\rho(x, \gamma y)\big] \tag{28}$$

where $c \in \mathbb{R}_+$, $s \in \mathcal{G}_S$ and $x, y \in X$.

Lemma 4.1 *For all $s \in \mathcal{G}_S$, $x, y, x_0, y_0 \in X$, $c \in \mathbb{R}$ and $\gamma_0 \in G$ we have*
(i) $P_c(\gamma_0 s, \gamma_0 x, y) = P_c(s, x, y)$
(ii) $P_c(s, x, y) \leq P_c(s, x_0, y_0)\, e^{\rho(x_0,x)+\rho(y_0,y)}$.
In particular the convergence of the series is independant of the choice of the points $x, y \in X$.

Proof. The equivariance property of the Poincaré series is straightforward. On the other hand by triangle inequality we have

$$\begin{aligned} P_c(s,x,y) &= \Sigma_{\gamma\in G} e^{-c d_S(s,\gamma)} \cosh\big[\rho(x,\gamma y)\big] \\ &\le \Sigma_{\gamma\in G} e^{-c d_S(s,\gamma)} \cosh\big[\rho(x_0,\gamma y_0)+\rho(x_0,x)+\rho(y_0,y)\big] \end{aligned}$$

hence we get

$$P_c(s,x,y) \le P_c(s,x_0,y_0)\,.e^{\rho(x_0,x)+\rho(y_0,y)}\,.$$

$\square$

The critical exponent of this series is defined as

$$c_0 =: \inf\{c>0 \quad / \quad P_c(s,x,y)<\infty\}\,.$$

Let x_0 be the point of X such that $L(S)=\max_i\{\rho(x_0,\sigma_i x_0)\}$. By the triangle inequality we have for all $\gamma\in\Gamma$, $\rho(x_0,\gamma x_0)\le L(S)\,l_S(\gamma)$, therefore

$$P_c(e,x_0,x_0) \le \Sigma_{\gamma\in\Gamma} e^{\big(c-L(S)\big)\,l_S(\gamma)}\,.$$

On the other hand, by definition of $\mathrm{Ent}_S\,\Gamma$, we have $\Sigma_{\gamma\in\Gamma} e^{-t\,l_S(\gamma)}<\infty$ for all $t>\mathrm{Ent}_S\,\Gamma$, hence we have proved that

$$c_0 \le \mathrm{Ent}_S\,\Gamma + L(S)\,. \tag{29}$$

We now consider untill the end of this section a $c\in\mathbb{R}_+$ such that $P_c(s,x,y)<\infty$.

Let us choose a probability measure μ with smooth density and compact support on X. For each $s\in\mathcal{G}_S$ let us define the measure on X

$$\mu_s^c = \Sigma_{\gamma\in G} e^{-c d_S(s,\gamma)} \gamma_*\mu \tag{30}$$

and the function $\mathcal{B}^c : G\times X\to\mathbb{R}$,

$$\mathcal{B}^c(s,x) = \int_X \cosh\big[\rho(x,z)\big]\,d\mu_s^c(z). \tag{31}$$

In the following lemmas 4.2, 4.3 and corollary 4.1 we show that $x\to\mathcal{B}^c(s,x)$ is a strictly convex C^2 function such that

$$\lim_{x\to\infty}\mathcal{B}^c(s,x) = +\infty\,.$$

Lemma 4.2 *Let c be such that $P_c(s,x,y)<\infty$. For all $s\in\mathcal{G}_S$ and $x\in X$, we have $\mathcal{B}^c(s,x)<\infty$. Moreover, the function $x\to\mathcal{B}^c(s,x)$ is stricly convex and $\lim_{x\to\infty}\mathcal{B}^c(s,x)=+\infty$.*

Proof. By definition of μ_s^c,

$$\mathcal{B}^c(s,x) = \int_X \Sigma_{\gamma\in G} e^{-c d_S(s,\gamma)} \cosh\big[\rho(x,\gamma z)\big]\,d\mu(z) = \int_X P_c(s,x,z)\,d\mu(z)\,,$$

so we get $\mathcal{B}^c(s,x) < \infty$ by lemma 4.1 (ii) since the support of μ is compact. For any geodesic $c(t)$ and z in X, $t \to d(c(t),z)$ is a convex function since (X,g) has negative sectional curvature, therefore $t \to \cosh\big[\rho(c(t),z)\big]$ is stricly convex and so is $x \to \mathcal{B}^c(s,x) = \int_X \cosh\big[\rho(x,z)\big] d\mu_s^c(z)$. On the other hand we have

$$\mathcal{B}^c(s,x) = \int_X \cosh\big[\rho(x,z)\big] d\mu_s^c(z) \geq \frac{1}{2} e^{\rho(x,x_0)} \int_X e^{-\rho(x_0,z)} d\mu_s^c(z)\,,$$

so $\mathcal{B}^c(s,x) \to +\infty$ whenever x tends to infinity in X. □

In the above lemma we proved that $x \to \mathcal{B}^c(s,x)$ is a convex function which tends to $+\infty$ when x tend to infinity. We shall now prove that $x \to \mathcal{B}^c(s,x)$ is a stricly convex C^2 function. We will also give estimates of the second derivative of $x \to \mathcal{B}^c(s,x)$.

Lemma 4.3 *Let c be such that $P_c(s,x,y) < \infty$. The function $x \to \mathcal{B}^c(s,x)$ is C^2 and for any $s \in \mathcal{G}_S$, $x \in X$ and any tangent vectors $v, w \in T_xX$ we have*

$$d\mathcal{B}^c(s,x)(v) = \int_X d\rho(x,z)(v) \sinh\big[\rho(x,z)\big] d\mu_s^c(z)$$

and

$$Dd\mathcal{B}^c(s,x)(v,w) =$$

$$\int_X \Big(\sinh\big[\rho(x,z)\big] Dd\rho(x,z)(v,w) + \cosh\big[\rho(x,z)\big] d\rho(x,z) \otimes d\rho(x,z)(v,w) \Big) d\mu_s^c(z)$$

Proof. Let $v \in T_xX$ be a unit tangent vector at a point $x \in X$. For each point $z \neq x$ in X, we have

$$d\big(\cosh\big[\rho(x,z)\big]\big)(v) = d\rho(x,z)(v)\, \sinh\big[\rho(x,z)\big]\,,$$

hence we get

$$|d\big(\cosh\big[\rho(x,z)\big]\big)(v)| = |d\rho(x,z)(v)\, \sinh\big[\rho(x,z)\big]| \leq \cosh\big[\rho(x,z)\big]\,, \tag{32}$$

therefore, $\cosh\big[\rho(x,z)\big] \leq 2\cosh\big[\rho(x_1,z)\big]$ for x in a sufficiently small neighbourhood of an arbitrary point x_1. Since $z \to 2\cosh\big[\rho(x_1,z)\big]$ is μ_s^c-integrable, we can differentiate $x \to \mathcal{B}^c(s,x)$ applying Lebesgue derivation theorem. Let us now compute the second derivative. Let $v, w \in T_xX$ be unit tangent vectors at $x \in X$. Let $\alpha(t)$ the geodesic such that $\alpha(0) = x$ and $\alpha'(0) = v$. We denote $W(t)$ the parallel vector field along α such that $W(0) = w$ and write $\rho_{(z,\alpha(t))}$ instead of $\rho(z,\alpha(t))$. Let us denote

$$h(t,z) = \frac{1}{t}\Big(d\rho_{(z,\alpha(t))}(W(t))\, \sinh\big[\rho_{(z,\alpha(t))}\big] - d\rho_{(z,\alpha(0))}(W(0))\, \sinh\big[\rho_{(z,\alpha(0))}\big]\Big).$$

When $z \neq x$ we have

$$\lim_{t\to 0} h(t,z) = \sinh\big[\rho_{(x,z)}\big] Dd\rho_{(x,z)}(v,w) + \cosh\big[\rho_{(x,z)}\big] d\rho_{(x,z)} \otimes d\rho_{(x,z)}(v,w)\,. \tag{33}$$

We will write

$$h_0(z) =: \lim_{t\to 0} h(t,z). \tag{34}$$

The formula which gives $Dd\mathcal{B}^c(s,x)(v,w)$ in lemma 4.3 is equivalent to

$$Dd\mathcal{B}^c(s,x)(v,w) = \int_X h_0(z)d\mu_s^c(z) \tag{35}$$

and will be a consequence of the dominated convergence theorem of Lebesgue with the existence of a $\mu_s^c(z)$- integrable function $H(z)$ such that for any $z \notin \alpha([0,t])$ then $h(t,z) \leq H(z)$. Let us now prove the existence of such a function H. For each $z \notin \alpha([0,t])$ we have

$$\begin{aligned}|h(t,z)| \quad \leq \quad & \sup_{t'\in[0,t]} \Big| \sinh\left[\rho_{(z,\alpha(t'))}\right] Dd\rho_{(z,\alpha(t'))}(\dot\alpha(t'),W(t')) + \dots \\ & \dots + \cosh\left[\rho_{(z,\alpha(t'))}\right] d\rho_{(z,\alpha(t'))} \otimes d\rho_{(z,\alpha(t'))}(\dot\alpha(t'),W(t'))\Big| .\end{aligned}$$

Since the curvature of (X,g) satisfies $-a^2 \leq K \leq -1$, the Rauch comparison theorem shows that for each $x,y \in X$

$$Dd\rho_{(x,y)} \leq a\frac{\cosh[a\rho_{(x,y)}]}{\sinh[a\rho_{(x,y)}]}\Big(g - d\rho_{(x,y)} \otimes d\rho_{(x,y)}\Big),$$

hence we get from the previous inequality

$$\begin{aligned}|h(t,z)| \quad \leq \quad & \Big[a\sinh\left[\rho_{(z,\alpha(t'))}\right]\frac{\cosh[a\rho_{(z,\alpha(t'))}]}{\sinh[a\rho_{(z,\alpha(t'))}]}\Big(g - d\rho_{(z,\alpha(t'))} \otimes d\rho_{(z,\alpha(t'))}\Big) + \dots \\ & \dots + \cosh\left[\rho_{(z,\alpha(t'))}\right] d\rho_{(z,\alpha(t'))} \otimes d\rho_{(z,\alpha(t'))}\Big](\dot\alpha(t'),W(t')) .\end{aligned}$$

But since $a \geq 1$ the concavity of the function tanh on $\mathbb{R}_+$ gives

$$\frac{a}{\tanh a\rho} \geq \frac{1}{\tanh \rho}$$

therefore we get

$$|h(t,z)| \quad \leq \quad a\sinh\left[\rho_{(z,\alpha(t'))}\right]\frac{\cosh[a\rho_{(z,\alpha(t'))}]}{\sinh[a\rho_{(z,\alpha(t'))}]} . \tag{36}$$

Since $\sinh\rho \leq \frac{1}{a}\sinh a\rho$ by convexity of sinh, we then get that $|h(t,z)| \leq H(z)$ from 36 for all $|t| \leq \frac{1}{a}$ and all $z \notin \alpha([0,t])$ where

$$H(z) = \begin{cases} a\frac{\cosh 1}{\sinh 1}\sinh[\rho_{(z,\alpha(0))}+1], & \rho_{(z,\alpha(0))} \geq \frac{2}{a} \\ \cosh[a\rho_{(z,\alpha(0))}+1], & \rho_{(z,\alpha(0))} < \frac{2}{a} \end{cases}$$

This concludes the proof of lemma 4.3. □

The above lemma 4.3 has the following

Corollary 4.1 *Under the assumptions of lemma 4.3 we have*

$$Dd\mathcal{B}^c \geq \mathcal{B}^c\,.g\,,$$

in particular, $\mathcal{B}^c$ is strictly convex.

Proof. Since the sectional curvature of (X,g) satisfies $K \leq -1$ Rauch's theorem shows that

$$Dd\rho \geq \frac{1}{\tanh\rho}\Big(g - d\rho\otimes d\rho\Big)\,.$$

From this inequality and lemma 4.3 we therefore get, for all $x \in X$ and any unit tangent vector $v \in T_xX$,

$$Dd\mathcal{B}^c(v,v) \geq \Big(\int_X \cosh[\rho_{(z,x)}]d\mu_s^c(z)\Big)g(v,v) = \mathcal{B}^c(x)\,.g(v,v)\,.$$

□

4.2 Construction of Lipschitzian maps $f_c : \mathcal{G}_S \to X$.

So far we have shown that for any $s \in \mathcal{G}_S$ the function $x \to \mathcal{B}^c(s,x)$ is strictly convex and tends to $+\infty$ when x tend to infinity. We then can define the map $f_c : \mathcal{G}_S \to X$ as follows. For $s \in \mathcal{G}_S$ we define $f_c(s)$ as the unique point $x \in X$ which achieves the strict minimum of the function $x \to \mathcal{B}^c(s,x)$. The end of this section is devoted to proving the following

Proposition 4.1 *Let c be such that $P_c(s,x,y) < \infty$. Let $f_c : (\mathcal{G}_S, d_S) \to (X,g)$ which associates to $s \in \mathcal{G}_S$ the unique point $x \in X$ which achieves the unique minimum of the function $x \to \mathcal{B}^c(s,x)$. Then, f_c is Lipschitzian of Lipschitz constant equal to c.*

The proof of the proposition 4.1 relies on the following technical lemmas.

Lemma 4.4 *Let c be such that $P_c(s,x,y) < \infty$. For all $x \in X$ and all tangent vector $v \in T_xX$ the function $\alpha : s \to d\mathcal{B}^c(s,x)(v)$ is differentiable at each point $s \in \mathcal{G}_S$ distinct from a vertex or a middle point of an edge. Moreover, for such an s we have*

$$\alpha'(s) \quad = \quad -c\int_X d\rho_{(x,z)}(v)\sinh\big[\rho(x,z)\big]\Sigma_{\gamma\in G}\frac{d}{ds}\Big(d_S(s,\gamma)\Big)e^{-cd_S(s,\gamma)}d(\gamma_*\mu)(z)$$

Proof. Let us denote by $[g,g']$ the edge containing s and parametrize it by $t \in [0,1]$. We first observe that for all $\gamma \in G$ then

$$d_S(s,\gamma) = \min\big[d_S(g,\gamma)+t, d_S(g',\gamma)+1-t\big]\,,$$

therefore $s \to d_S(s,\gamma)$ is differentiable at each $s \in]g,g'[$ distinct of the middle point of $]g,g'[$. On the other hand we have by lemma 4.3

$$d\mathcal{B}^c(s,x)(v) = \int_X d\rho(x,z)(v)\sinh\big[\rho(x,z)\big]d\mu_s^c(z)\,,$$

so that we can write

$$\tfrac{1}{t}\Big(\alpha(s+t)-\alpha(s)\Big) =$$
$$\Sigma_{\gamma\in G}\int_X d\rho_{(x,\gamma z)}(v)\sinh\big[\rho(x,\gamma z)\big]\,.\tfrac{1}{t}\Big[e^{-cd_S(s+t,\gamma)}-e^{-cd_S(s,\gamma)}\Big]d\mu(z) \quad ,$$

where we have identified the point s in the edge $[g,g']$ with its parameter. Let us observe that for $|t|$ small enough,

$$\Big|\frac{1}{t}\Big[e^{-cd_S(s+t,\gamma)}-e^{-cd_S(s,\gamma)}\Big]\Big| \leq 2c\,e^{-cd_S(s,\gamma)}\,,$$

and that

$$2c\,\Sigma_{\gamma\in G}\int_X \big|d\rho_{(x,\gamma z)}(v)\big|\sinh\big[\rho(x,\gamma z)\big]\,.e^{-cd_S(s,\gamma)}d\mu(z)<\infty\,,$$

hence if $s\in\mathcal{G}_S$ is distinct from a vertex or a middle point of an edge we get

$$\lim_{t\to 0}\tfrac{1}{t}\Big(\alpha(s+t)-\alpha(s)\Big) =$$
$$-c\int_X d\rho_{(x,z)}(v)\sinh\big[\rho(x,z)\big]\Sigma_{\gamma\in G}\tfrac{d}{ds}\Big(d_S(s,\gamma)\Big)e^{-cd_S(s,\gamma)}d(\gamma_*\mu)(z)$$

by Lebesgue's theorem. □

Lemma 4.5 *Let c be such that $P_c(s,x,y)<\infty$. Let $s_0\in\mathcal{G}_S$ be a point distinct from a vertex or a middle point of an edge, and u a unit vector tangent at s_0 to the edge containing s_0. Then, we have $||df_c(u)||\leq c$.*

Proof. Let us fix a smooth moving frame $\{E_1,\ldots,E_n\}$ of TX and define the function $\Phi : X\times\mathcal{G}_S\to\mathbb{R}^n$ by

$$\Phi(x,s)=\big(d\mathcal{B}^c(s,x)(E_1),\ldots,d\mathcal{B}^c(s,x)(E_n)\big)\,.$$

By definition, the point $f_c(s)$ is characterized by the implicit equation

$$\Phi(f_c(s),s)=0\,,$$

or equivalently,

$$d\mathcal{B}^c(s,f_c(s))=0\,.$$

For all $x\in X$ and $s\in\mathcal{G}_S$ in a neighbourhood of s_0 the function Φ is differentiable by lemma 4.3 and 4.4. Moreover since $x=f_c(s)$ is a critical point of the function $x\to\mathcal{B}^c(s,x)$, we have, for $j=1,\ldots,n$,

$$\frac{\partial\Phi}{\partial x}(f_c(s),s)(E_j)=\big(Dd\mathcal{B}^c(s,f_c(s))(E_j,E_1),\ldots,Dd\mathcal{B}^c(s,f_c(s))(E_j,E_n)\big)\,,$$

thus $\frac{\partial\Phi}{\partial x}(f_c(s),s)$ is invertible by corollary 4.1. By the implicit function theorem, the function f_c is then differentiable at s in a neighbourhood of s_0 and we have, if u is a unit vector tangent at s_0 to the edge containing s_0 and v a tangent vector in $T_{f_c(s)}X$,

$$Dd\mathcal{B}^c(s_0, f_c(s_0))(df_c(u), v)) = -\frac{d}{ds}_{/s=s_0} d\mathcal{B}^c(s, f_c(s_0))(v)\,. \tag{37}$$

From corollary 4.1 and lemma 4.4 we obtain, setting $v = \frac{df_c(u)}{||df_c(u)||}$

$$g(df_c(u), v)\,\mathcal{B}^c(s_0, f_c(s_0)) \leq$$
$$c\int_X \left|d\rho_{(f_c(s_0),z)}(v)\right| \sinh\left[\rho(f_c(s_0), z)\right] \Sigma_{\gamma\in G}\left|\tfrac{d}{ds}_{/s=s_0}\Big(d_S(s,\gamma)\Big)\right| e^{-c d_S(s_0,\gamma)} d(\gamma_*\mu)(z)$$

therefore

$$\left|g(df_c(u), v)\,\mathcal{B}^c\big(s_0, f_c(s_0)\big)\right| \leq c\int_X \sinh\left[\rho(f_c(s_0), z)\right] d\mu^c_{s_0}(z)\,. \tag{38}$$

hence

$$||df_c(u)|| \leq c\,\frac{\int_X \sinh\left[\rho(f_c(s_0), z)\right] d\mu^c_{s_0}(z)}{\int_X \cosh\left[\rho(f_c(s_0), z)\right] d\mu^c_{s_0}(z)} \leq c\,,$$

which completes the proof of lemma 4.5. □

The proposition 4.1 follows then from the

Corollary 4.2 *Let c be such that $P_c(s,x,y) < \infty$. The map f_c is Lipschitzian of Lipschitz constant equal to c.*

Proof. Let us consider a segment $[s_1, s_2] \subset \mathcal{G}_S$ which contains no edges nor middle points. It directly follows from lemma 4.5 that

$$\rho\big(f_c(s_1), f_c(s_2)\big) \leq c\, d_S(s_1, s_2)\,. \tag{39}$$

We now want to extend the inequality (39) for all points $s_1, s_2 \in \mathcal{G}_S$. For that purpose we first consider a segment $[s_1, s_2] \subset \mathcal{G}_S$ where s_1 is a midpoint of an edge e and s_2 a vertex of the same edge e and the inequality (39) for these points s_1, s_2 derives from the continuity of f_c at x_1 and x_2. The corollary 4.2 will then follow from the fact that any segment $[s_1, s_2] \subset \mathcal{G}_S$ can be decomposed in a finite sequence of adjacent intervals $[y_1^k, y_2^k]$ where y_1^k is a midpoint and y_2^k a vertex of one same edge or the other way around. Let us prove the continuity of f_c at a vertex or a midpoint s of an edge. Given such a point s, let $\{s_k\}_{k\in\mathbb{N}}$ be a sequence converging to s and staying in a single mid-edge containing s. The sequence $x_k =: f_c(s_k)$ is a Cauchy sequence in X by (39) whose limit is a point $x = \lim_k x_k$. We want to prove that $f_c(s) = x$. For all $z \in X$ and $k \in \mathbb{N}$ we have

$$\mathcal{B}^c(s_k, z) \geq \mathcal{B}^c(s_k, x_k) \tag{40}$$

by definition of $x_k = f_c(s_k)$. We claim that $\lim_k \mathcal{B}^c(s_k, x_k) = \mathcal{B}^c(s, x)$ and that $\lim_k \mathcal{B}^c(s_k, z) = \mathcal{B}^c(s, z)$. Assuming the claim and passing to the limit in (40) when k tends to infinity gives for all $z \in X$,

$$\mathcal{B}^c(s, z) \geq \mathcal{B}^c(s, x)\,, \tag{41}$$

therefore $x = f_c(s)$. Let us prove the claim. By definition (31) and (30), we have

$$\begin{aligned} \mathcal{B}^c(s_k, x_k) &= \int_X \cosh\left[\rho(x_k, z)\right] d\mu^c_{s_k}(z) \\ &= \int_X \Sigma_{\gamma\in G} e^{-cd_S(s_k,\gamma)} \cosh\left[\rho(x_k, \gamma z)\right] d\mu(z)\,, \end{aligned}$$

Since $e^{-cd_S(s_k,\gamma)} \cosh\left[\rho(x_k, \gamma z)\right] \leq e^c\, e^{-cd_S(s,\gamma)} \cosh\left[\rho(x, \gamma z) + 1\right]$ for k large enough, we get $\lim_k \mathcal{B}^c(s_k, x_k) = \mathcal{B}^c(s, x)$ by the Lebesgue's theorem. By the way we get $\lim_k \mathcal{B}^c(s_k, z) = \mathcal{B}^c(s, z)$ which concludes the proof of the claim, the corollary 4.2 and the proposition 4.1. □

5 Algebraic Entropy and η-straight isometries.

Let G be a finitely generated discrete group of isometries of (X, g) whose sectional curvature satisfies $-a^2 \leq K_g \leq -1$, and $S = \{\sigma_1, \ldots, \sigma_p\}$ be a finite generating set.

We assume that the minimal displacement $L(S) = \inf_{x\in X} \max_{i=1,\ldots p} \rho(x, \sigma_i x)$ of S (cf. definition 2.1) satisfies $L(S) > 0$. By lemma 3.1 there exist a point $x_0 \in X$ such that

$$L(S) = \inf_{x\in X} \max_{i\in\{1,\ldots,p\}} \rho(x, \sigma_i x) = \max_{i\in\{1,\ldots,p\}} \rho(x_0, \sigma_i x_0)\,.$$

The goal of this section is to prove that if all elements of G are "almost non η-straight" for some η such that $L(S) > \eta > 0$, then the entropy of G with respect to S is bounded below by η. By "almost non η-straight" elements we mean isometries γ such that $\rho(x_0, \gamma x_0) \leq (L(S) - \eta) l_S(\gamma) + D$, for some positive number D.

Theorem 5.1 *Let G be a finitely generated discrete group of isometries of (X, g) whose sectional curvature satisfies $-a^2 \leq K_g \leq -1$, and $S = \{\sigma_1, \ldots, \sigma_p\}$ be a finite generating set of G with $L(S) = \inf_{x\in X} \max_{i\in\{1,\ldots,p\}} \rho(x, \sigma_i x) = \max_{i\in\{1,\ldots,p\}} \rho(x_0, \sigma_i x_0) > 0$. Let us assume that there exist $D \geq 0$ and η, $0 < \eta < L(S)$, such that for all $\gamma \in G$,*

$$\rho(x_0, \gamma x_0) \leq (L(S) - \eta) l_S(\gamma) + D\,, \tag{42}$$

then $\mathrm{Ent}_S(G) \geq \eta$.

Proof. The proof relies on the construction made in section 3 of an equivariant Lipschitzian map of Lipschitz constant $c > \mathrm{Ent}_S(G) + L(S) - \eta$.

Let us prove that under the assumption (42) then for any $c > \mathrm{Ent}_S(G) + L(S) - \eta$ we have $P_c(s, x, y) < \infty$. By triangle inequality we have

$$e^{-cd_S(s,\gamma)} \leq e^{cd_S(s,e)} e^{-cd_S(\gamma,e)}\,,$$

and for any $x_0 \in X$

$$\cosh\left[\rho(x,\gamma y)\right] \le e^{\rho(x,\gamma y)} \le e^{\rho(x,x_0)+\rho(x_0,y)+\rho(x_0,\gamma x_0)}\,.$$

Therefore for x_0, D and η chosen such that (42) holds, we get

$$P_c(s,x,y) \le e^{D+c\,d_S(e,s)+\rho(x,x_0)+\rho(x_0,y)}\Sigma_{\gamma\in G}e^{[L(S)-\eta-c]d_S(e,\gamma)}\,,$$

and so $P_c(s,x,y)<\infty$ for each $c>\mathrm{Ent}_S(G)+L(S)-\eta$.

Hence by section 3, proposition 4.1 there exists an equivariant Lipschitzian map $f_c:(\mathcal{G}_S,d_S)\to(X,g)$ of Lipschitz constant c for any $c>\mathrm{Ent}_S(G)+L(S)-\eta$. We consider the point $x=f_c(e)$, where e is the neutral element of G. By definition of $L(S)$, there is a $\sigma_i\in S$ such that $\rho(x,\sigma_i x)\ge L(S)$. Therefore, by equivariance,

$$\rho\big(f_c(e),\sigma_i(f_c(e))\big)=\rho\big(f_c(e),f_c(\sigma_i(e))\big)\ge L(S)\,.$$

On the other hand, since f_c is c-Lipschitzian we have

$$\rho\big(f_c(e),f_c(\sigma_i(e))\big)\le c\,d_S(e,\sigma_i(e))=c\,.$$

The two above inequalities give

$$c\ge L(S)$$

and since c is any number such that $c>\mathrm{Ent}_S(G)+L(S)-\eta$, we get $\mathrm{Ent}_S(G)\ge\eta$.
□

6 Proof of the main theorem.

In this section we shall first prove that the entropy of a group with respect to a set of two generators with displacement $L>0$ is bounded below. Then we shall prove the main theorem.

Proposition 6.1 *Let (X,g) be a Cartan-Hadamard manifold whose sectional curvature satisfies $a^2\le K_g\le -1$ and G a discrete group of isometries of (X,g) generated by two isometries $\{\sigma_1,\sigma_2\}$. Let us assume*

$$L=\inf_{x\in X}\max\{\rho(x,\sigma_1 x),\rho(x,\sigma_2 x)\}>0\,.$$

Then the entropy of G relatively to the set of generators $\Sigma=\{\sigma_1,\sigma_2\}$ satisfies

$$ent_\Sigma G\ge\min\left[\frac{\log(\cosh(\frac{L}{4}))}{5+\log(\cosh(\frac{L}{4}))}\frac{\log 2}{6},\frac{1}{1000}\Big(1-\frac{\cosh(\frac{L}{4})}{\cosh(\frac{L}{2})}\Big)^4\right].$$

Proof. Let $\delta=\log\cosh(\frac{L}{4})$. The proof divides into two cases. In the first case we can find two elements in G of bounded length l_Σ which are hyperbolic with distinct axes and displacement larger than δ. In that case, a ping-pong argument shows that the semigroup generated by these two elements (or their inverse) is free with corresponding entropy bounded below by a constant depending on δ. In the second case, when we cannot find such a free semigroup, then we can show that all elements of G are almost non η-straight for some $\eta = \eta(\delta, L)$ and we will conclude by theorem 5.1.

Case 1. There exists an element $\gamma \in G$ of algebraic length $l_\Sigma(\gamma) \leq 4$ whose displacement $l(\gamma)$ in X is bounded below $l(\gamma) > \delta$.

Case 2. The displacement of all elements $\gamma \in G$ of algebraic length $l_\Sigma(\gamma) \leq 4$ satisfies $l(\gamma) \leq \delta$.

In the case 1, let us consider an element $\gamma \in G$ of algebraic length $l_\Sigma(\gamma) \leq 4$ and whose displacement $l(\gamma)$ in X satisfies $l(\gamma) > \delta$. We note that γ is then an hyperbolic isometry of X. Since G is not virtually nilpotent one of the generators σ_1 or σ_2, say σ_1 does not preserve the axis of γ. Indeed if both σ_1 and σ_2 were preserving the axis of γ, then G would preserve the axis of γ and hence would be virtually abelian by lemma 2.2 (ii), contradiction. Then, if (θ, η) are the endpoints of the axis of γ, $\sigma_1(\{\theta, \eta\}) \cap \{\theta, \eta\} = \emptyset$ by the proof of lemma 2.1, a). We can then apply the effective ping-pong lemma proved in the appendix to the two hyperbolic elements γ and $\sigma_1\gamma\sigma_1^{-1}$ which have disjoint fixed-point-sets. This shows that the algebraic entropy of the subgroup generated by γ and $\sigma_1\gamma\sigma_1^{-1}$ is bounded below by $\frac{\delta}{5+\delta}\log 2$. We then deduce that,

$$\mathrm{Ent}_\Sigma(\Gamma) \geq \frac{\delta}{5+\delta}\frac{\log 2}{6}\,.$$

In the case 2, proposition 3.2 tells that all elements $\gamma \in G$ of length $l_\Sigma(\gamma) = 6$ are not (L, η)-straight where η is given by (23), $\eta = 10^{-3}\Big(1 - \frac{\cosh(\frac{L}{4})}{\cosh(\frac{L}{2})}\Big)^4$. Then every element $g \in \Gamma$ of algebraic length 6 satisfies,

$$\rho(x_0, gx_0) \leq (L - \eta) l_\Sigma(g)\,.$$

Hence, one obtain that every element $\gamma \in \Gamma$, satisfies,

$$\rho(x_0, \gamma x_0) \leq (L - \eta)(l_\Sigma(\gamma) - 5) + 5L\,.$$

Therefore we get from theorem 5.1 that $\mathrm{Ent}_\Sigma G \geq \eta = 10^{-3}\Big(1 - \frac{\cosh(\frac{L}{4})}{\cosh(\frac{L}{2})}\Big)^4$. $\square$

We may now prove the main theorem which we recall below,

Theorem 6.1 (Main theorem) *Let (X, g) be a Cartan-Hadamard manifold whose sectional curvature satisfies $-a^2 \leq K_g \leq -1$. Let Γ be a discrete and finitely generated subgroup of the isometry group of (X, g), then either Γ is virtually nilpotent or its algebraic entropy is bounded below by an explicit constant $C(n, a)$.*

Remark. The constant is

$$C(n,a) = \min\Bigg[\frac{\log(\cosh(\frac{\mu(n,a)}{4}))}{5 + \log(\cosh(\frac{\mu(n,a)}{4}))}\frac{\log 2}{12}, \frac{1}{2000}\Big(1 - \frac{\cosh(\frac{\mu(n,a)}{4})}{\cosh(\frac{\mu(n,a)}{2})}\Big)^4, \frac{\mu(n,a)}{4},$$

$$\frac{1}{4}\Big(1-\frac{1}{\cosh\mu(n,a)}\Big)^2, \frac{1}{2}\log\left(\frac{1}{2}\Big(\cosh\frac{\mu(n,a)}{2}+\frac{1}{\cosh\frac{\mu(n,a)}{2}}\Big)\right)\Bigg].$$

Proof. If $S = \{\sigma_1, \ldots, \sigma_p\}$ is a finite generating set of Γ, proposition 2.1 allows to reduce to the following three cases,

i) there exist $\sigma_i, \sigma_j \in S$ such that $L(<\sigma_i, \sigma_j>) \geq \mu(n,a)$ and such that the subgroup $<\sigma_i, \sigma_j>$ is not virtually nilpotent.

ii) There exist $\sigma_i, \sigma_j, \sigma_k \in S$ such that $L(<\sigma_i\sigma_j, \sigma_k>) \geq \mu(n,a)$ and such that $<\sigma_i\sigma_j, \sigma_k>$ is not virtually nilpotent.

iii) All σ_i's are elliptic and, for all $i \neq j$, the subgroup $<\sigma_i, \sigma_j>$ fixes a point $y \in X$ or a point $\theta \in \partial X$.

In the first case (resp. the second case) proposition 6.1 gives a lower bound of the algebraic entropy of $<\sigma_i, \sigma_j>$ (resp. $<\sigma_i\sigma_j, \sigma_k>$) with respect to the generating set $\{\sigma_i, \sigma_j\}$ (resp. $\{\sigma_i\sigma_j, \sigma_k\}$), by the number,

$$\min\left[\frac{\log(\cosh(\frac{\mu(n,a)}{4}))}{5+\log(\cosh(\frac{\mu(n,a)}{4}))}\frac{\log 2}{6}, \frac{1}{1000}\Big(1-\frac{\cosh(\frac{\mu(n,a)}{4})}{\cosh(\frac{\mu(n,a)}{2})}\Big)^4\right],$$

using the fact that $L(\sigma_i, \sigma_j) \geq \mu(n,a)$, (resp. $L(\sigma_i\sigma_j, \sigma_k) \geq \mu(n,a)$). We conclude in the two first cases (i) and (ii) by noticing that the entropy of Γ with respect to S is bounded below by $\mathrm{Ent}_{\{\sigma_i,\sigma_j\}}(<\sigma_i, \sigma_j>)$ (resp. by $\frac{1}{2}\mathrm{Ent}_{\{\sigma_i\sigma_j,\sigma_k\}}(<\sigma_i\sigma_j, \sigma_k>)$), since $d_{\{\sigma_i,\sigma_j\}} \geq d_S$ (resp. $d_{\{\sigma_i\sigma_j,\sigma_k\}} \geq \frac{1}{2}d_S$).

In the third case, proposition 3.1, implies that,

$$\rho(x_0, \gamma x_0) \leq (L(S) - \eta/2)(l_S(\gamma) - 1) + L(S),$$

where η is given in proposition 3.1. We conclude by applying theorem 5.1, which gives $\mathrm{Ent}_S(\Gamma) \geq \eta$, and then bounding below η using $L(S) \geq \mu(n,a)$. □

7 Appendix

In this section (X, g) is a Cartan-Hadamard manifold of sectional curvature $K \leq -1$. It is well known that if α, β are two hyperbolic isometries of (X, g) with disjoint axes, then a sufficiently large power α^N and β^N of α and β generates a non abelian free group of $\mathrm{Isom}\, X$. In [6], [8], it was shown that if Γ is an hyperbolic group then N can be chosen independantly of α and β in Γ and under the same assumptions the N was shown to depend only on the number of generators and the constant of hyperbolicity of Γ [6]. In what follows we show that $N = N(\delta)$ can be chosen independantly of α and β two hyperbolic isometries of (X, g) with disjoint set of fixed points and displacement greater than or equal to a positive number δ.

Proposition 7.1 *Let (X, g) be a Cartan Hadamard manifold of sectional curvature $K \leq -1$ and Γ a discrete group of isometries in $Isom(X, g)$. We assume that α and β have disjoint set of fixed point and their displacement satisfy $l(\alpha) \geq \delta$ and $l(\beta) \geq \delta$, where δ is a positive number. Then, (α^N, β^N) or (α^N, β^{-N}) generates a non abelian free semi-group, where $N = E(\frac{5}{\delta}) + 1$.*

Before going to the proof of the proposition 7.1 let us set some notations. Let us write $x = x(t)$ and $y = y(t)$, $t \in \mathbb{R}$, the axes of α and β. The points $\theta^{\pm} = \lim_{t\to\pm\infty} x(t)$ and $\zeta^{\pm} = \lim_{t\to\pm\infty} y(t)$ are the fixed points of α and β on the ideal boundary ∂X of X. Let us denote x^+ and x^- the projections of ζ^+ and ζ^- on the axis of α. We can assume that x^+ is closer to θ^+ than x^-, (if it is not the case, we replace β by β^{-1}). Let us also denote y_0 the projection of x^+ on the axis of β. We now parametrize x and y in such a way that $x(0) = x^+$ and $y(0) = y_0$. We set $t_1 = Nl(\alpha) = l(\alpha^N)$ and $t_2 = Nl(\beta) = l(\beta^N)$, where $N = E(\frac{5}{\delta}) + 1$ is chosen as in the proposition. We define $U^{\pm}$ as the set of points $p \in X$ such that $\rho(p, x(\pm t_1)) \leq \rho(p, x(o))$. In the same way we define $V^{\pm}$ as the set of points $p \in X$ such that $\rho(p, y(\pm t_2)) \leq \rho(p, y(o))$. For a unit tangent vector $u \in T_xX$ at a point $x \in X$ and $\alpha \in [0, \pi[$ we denote $\mathcal{C}(u, \alpha) = \{exp_x v : v \in T_xX\, , \angle(u, v) \in [0, \alpha[\,\}$ the cone of angle α at x, where exp_x is the exponential map at x.

We further need the following geometric lemmas. For a triangle ABC in (X, g), we will write $\hat{A}$ the angle at A, and a, b, c the length of the sides opposite to A, B and C.

Lemma 7.1 *Let ABC be a triangle in (X, g) such that $\frac{\pi}{6} \leq \hat{A} \leq \pi$, then $\rho(B, C) > \rho(A, B) + \rho(A, C) - 4$. Moreover, if $\hat{A} \geq \frac{\pi}{2}$, then $\rho(B, C) > \rho(A, B) + \rho(A, C) - 1$.*

Proof. Since the curvature $K \leq -1$, we have

$$\cosh a \geq \cosh b \, \cosh c \, - \cos \hat{A} \, \sinh b \, \sinh c. \tag{43}$$

The first inequality of lemma 7.1 will therefore be a consequence of the fact that if $b + c > 4$ then

$$\cosh(b + c - 4) - \cosh b \, \cosh c \, + \cos \hat{A} \, \sinh b \, \sinh c < 0. \tag{44}$$

Setting $X = e^{-(b+c)}$ we have

$$\cosh(b + c - 4) - \cosh b \, \cosh c \, + \cos \hat{A} \, \sinh b \, \sinh c =$$

$$e^{(b+c)}\Big[(2e^4 - 1 + \cos \hat{A})X^2 - (e^{-2b} + e^{-2c})(1 + \cos \hat{A}) - (1 - \cos \hat{A} - 2e^{-4})\Big].$$

Since $e^{-2b} + e^{-2c} \geq 2e^{-(b+c)}$, we then get

$$\cosh(b + c - 4) - \cosh b \, \cosh c \, + \cos \hat{A} \, \sinh b \, \sinh c \leq e^{(b+c)} P(X)$$

where

$$P(X) = (2e^4 - 1 + \cos \hat{A})X^2 - (1 + \cos \hat{A})X - (1 - \cos \hat{A} - 2e^{-4})$$

and $P(X)$ is negative when $P(0) < 0$ and $P(e^{-4}) < 0$ which is the case if $\cos \hat{A} < 1 - 2e^{-4}$ and so when $\hat{A} \geq \pi/6$. This proves the first inequality of the lemma. The second inequality is proved similarly when $\cos \hat{A} < 1 - 2e^{-1}$. □

Lemma 7.2 *The sets U^+ and U^- are contained in $\mathcal{C}(\dot{x}(0), \pi/6)$ and $\mathcal{C}(-\dot{x}(0), \pi/6)$ respectively.*

Proof. We recall that $x(0) = x^+$. Let $c(t)$ be a geodesic ray starting at x^+ such that $\angle(\dot{x}(0), \dot{c}(0)) \geq \pi/6$. Since $t_1 \geq 5$, the lemma 7.1 implies

$$\rho(c(t), x(t_1)) > \rho(x^+, c(t)) + \rho(x^+, x(t_1)) - 4 \geq \rho(c(t), x^+)$$

therefore $c(t) \notin U^+$. The same argument holds for U^-. □

Let us denote by z_t the geodesic joining x^+ and $y(t)$ and $z_{\pm\infty}$ the geodesic joining x^+ and $y(\pm\infty) = \zeta^{\pm\infty}$.

Lemma 7.3 *The set $V^\pm$ is contained in $\mathcal{C}(\dot{z}_{\pm\infty}(0), \pi/3)$.*

Proof. Let us recall that the angle at $y(0) = y_0$ between z_0 and y is equal to $\pi/2$, so that the lemma 7.1 says that

$$\mathrm{length(z_t) > length(z_0) + t - 1}\,, \tag{45}$$

and in particular,

$$\mathrm{length(z_{t_2}) > length(z_0) + t_2 - 1}\,. \tag{46}$$

Let us now show that $\angle(\dot{z}_{t_2}(0), \dot{z}_{+\infty}(0)) \leq \pi/6$. Assume by contradiction that $\angle(\dot{z}_{t_2}(0), \dot{z}_{+\infty}(0)) > \pi/6$, then by lemma 7.1 we have, when t tends to $+\infty$,

$$t - t_2 > \mathrm{length(z_{t_2}) + length(z_t)} - 4\,, \tag{47}$$

but summing up (45) and (46), in (47) leads to a contradiction since $t_2 \geq 5$. Therefore we have $\angle(\dot{z}_{t_2}(0), \dot{z}_{+\infty}(0)) \leq \pi/6$. Let now consider a geodesic ray c starting at x^+ such that $\angle(\dot{c}(0), \dot{z}_{+\infty}(0)) \geq \pi/3$. Thus, $\angle(\dot{c}(0), \dot{z}_{t_2}(0)) \geq \pi/6$ and by lemma 7.1 we get

$$\rho(c(t), y(t_2)) > \rho(c(t), x^+) + \mathrm{length(z_{t_2})} - 4\,,$$

and applying again the lemma 7.1,

$$\rho(c(t), y(t_2)) > \rho(c(t), x^+) + \mathrm{length(z_0) + t_2} - 5\,.$$

The last inequality becomes by triangle inequality,

$$\rho(c(t), y(t_2)) > \rho(c(t), y_0) + t_2 - 5\,,$$

therefore $\rho(c(t), y(t_2)) > \rho(c(t), y_0)$ since $t_2 \geq 5$.

We have proved that a geodesic ray c starting at x^+ such that $\angle(\dot{c}(0), \dot{z}_{+\infty}(0)) \geq \pi/3$ does not intersect V^+. This proves that $V^+ \subset \mathcal{C}(\dot{z}_{+\infty}(0), \pi/3)$. By the same argument we also have $V^- \subset \mathcal{C}(\dot{z}_{-\infty}(0), \pi/3)$, which ends the proof of the lemma. □

Lemma 7.4 *We have $U^\pm \cap V^\pm = \emptyset$, $U^+ \cap U^- = \emptyset$, $V^+ \cap V^- = \emptyset$.*

Proof. From the angle relations, $\angle(\dot{x}(0), \dot{z}_{+\infty}(0)) = \pi/2$, $\angle(\dot{x}(0), \dot{z}_{-\infty}(0)) \geq \pi/2$, $\angle(\dot{x}(0), -\dot{x}(0)) = \pi$, and the relative position of x^+, x^- and θ^+, it follows that $\mathcal{C}(\dot{x}(0), \pi/6)$ does not intersect $\mathcal{C}(\dot{z}_{+\infty}(0), \pi/3)$, $\mathcal{C}(-\dot{x}(0), \pi/6)$ and $\mathcal{C}(\dot{z}_{-\infty}(0), \pi/3)$. Therefore by the lemmas 7.2, 7.3 we conclude that U^+ does not intersect U^-, V^+ and V^-. Now since $\angle(-\dot{x}(0), \dot{z}_{+\infty}(0)) = \pi/2$, we have $\mathcal{C}(\dot{z}_{+\infty}(0), \pi/3) \cap \mathcal{C}(-\dot{x}(0), \pi/6) = \emptyset$, hence $V^+ \cap U^- = \emptyset$. If $p \in V^+ \cap V^-$, we have $\rho(p, y(0)) \geq \rho(p, y(-t_2))$ and $\rho(p, y(0)) \geq \rho(p, y(t_2))$ which contradicts the convexity of the function $t \to \rho(x(t), p)$. Therefore $V^+ \cap V^- = \emptyset$. $\square$

Lemma 7.5 *We have $\alpha^N(V^+) \subset U^+$ and $\beta^N(U^+) \subset V^+$.*

Proof. Since x and y are the axes of α^N and β^N respectively we have $\alpha^N(x(-t_1)) = x(0)$, $\beta^N(y(-t_1)) = y(0)$, $\alpha^N(x(0)) = x(t_1)$ and $\beta^N(y(0)) = y(t_2)$. Therefore for any $p \in X - U^-$, we have $\alpha^N(p) \in U^+$ and similarly for any $p \in X - V^-$, we have $\beta^N(p) \in V^+$ by definition of N. On the other hand, by the lemma 7.4, we have $V^+ \subset X - U^-$ and $U^+ \subset X - V^-$, which concludes. $\square$

The proof of the proposition 7.1 is a direct application of the lemma 7.5 by a standard ping-pong argument.

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